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## On Urabe's criteria of isochronicity

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**Abstract.** We give a short proof of Urabe's criteria for the isochronicity of periodical solutions of the equation  $\ddot{x} + g(x) = 0$ . We show that apart from the harmonic oscillator there exists a large family of isochronous potentials which must be all non-polynomial and not symmetric (even function of the coordinate  $x$ ).

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We consider a system of differential equations of the form

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -g(x),\end{aligned}\tag{1}$$

where we suppose

$$g(x) \in C(a, b), \quad xg(x) > 0 \text{ for } x \neq 0, g(0) = 0 \text{ and } g'(0) = k \neq 0.\tag{2}$$

Denoting

$$U(x) = \int_0^x g(s)ds$$

we obtain the first integral in the form "kinetic energy+potential energy", i.e. in the form

$$H(x, y) \stackrel{\text{def}}{=} \frac{y^2}{2} + U(x) = E,\tag{3}$$

such that  $H(x, y)$  is the Hamiltonian and (1) are the Hamilton equations of the motion of our system [3].

It is well known that any solution near the origin oscillates around  $x = 0, y = 0$  with a bounded period, i.e. system (1) has a center in the origin. The problem arises then to determine whether the period of oscillations is constant for all solutions near the origin. A center with such property is called *isochronous*. At present the problem of isochronicity is of renewed interest (see, for example, [2] for current references).

It was shown in [1] that if  $g(x)$  is a polynomial, then system (1) cannot have an isochronous center, except when  $g(x)$  is linear  $g(x) = kx$ , in which case  $k = (2\pi/\tau)^2$ , where  $\tau$  is the period of oscillations. If  $g(x)$  is not exactly linear, then still the period of oscillations infinitesimally close to the origin is also equal to  $\tau$ .

In the present Letter we give a simple short proof of the following Urabe's criteria [7] of isochronicity of the center of system (1).

**Theorem 1** *When  $g(x)$  is continuous, the necessary and sufficient condition that  $g(x) \in C^1(a, b)$  and system (1) has an isochronous center in the origin, is that, in the neighbourhood of  $x = y = 0$  by the transformation*

$$\frac{1}{2}X^2 = U(x),\tag{4}$$

where  $X/x > 0$  for  $x \neq 0$ ,  $g(x)$  is expressed as

$$g(x) = g[x(X)] \stackrel{\text{def}}{=} h(X) = \frac{2\pi}{\tau} \frac{X}{1 + S(X)},\tag{5}$$

where  $S(X)$  is an arbitrary continuous odd function and  $\tau$  is the period of the oscillations.

First in [7] Urabe proved the criteria in the case when  $g(x)$  is an analytic function. For function  $g(x) \in C^1$  he got a more complicated criteria with the function  $h(X)$  of the form

$$h(X) = \frac{2\pi}{\tau} \frac{X}{1 + S(X) + R(X)},$$

where  $S(X)$  is an odd and  $R(X)$  is an even continuous function (see [7]). Then in [8] he showed that if  $g(x) \in C^1(a, b)$  then necessarily  $R(X) \equiv 0$ .

Note that in the statement of the theorem Urabe demands the additional property

$$S(0) = 0, \quad XS(X) \in C^1,$$

but every continuous odd function has the property  $S(0) = 0$ , and the second one is not essential for our proof. We have also required  $g(x)$  to be smooth in a neighbourhood of  $x = 0$  (as in the original work by Urabe [7], but in fact it is sufficient for our reasoning if  $g(x)$  is continuous in a neighbourhood of the origin and differentiable at  $x = 0$ ).

Our proof of the Theorem 1 is based on the following criteria, which for the first time appears, apparently, in Landau and Pyatigorsky [4] and which later was rederived by Keller [5, 6] (who also considered some connected problems, in particular, the case of non-monotonic potential). For convenience of the reader we present the criteria with the proof, which stems from the books [3, 4], here.

**Theorem 2** *When  $g(x)$  is continuous and the conditions (2) hold, system (1) has an isochronous center of the period  $\tau$  at the origin if and only if*

$$x_2(U) - x_1(U) = \frac{\sqrt{2}\tau}{\pi} \sqrt{U}, \quad (6)$$

for  $U \in (0, U_0)$ , where  $x_1(U)$  is the inverse function to  $U(x)$  for  $x \in (a, 0)$  and  $x_2(U)$  is the inverse function to  $U(x)$  for  $x \in (0, b)$ .

*Proof.* First we note that due to (2) the functions  $x_1(U), x_2(U)$  are defined and  $x_1(U), x_2(U) \in C^1(0, U_0)$  with a  $U_0 > 0$ . Denote by  $T(E)$  the period of the orbit of (1) corresponding to the value of energy  $E$ . Then we have [3]

$$T(E) = \sqrt{2} \int_0^E \left[ \frac{dx_2(U)}{dU} - \frac{dx_1(U)}{dU} \right] \frac{dU}{\sqrt{E - U}}. \quad (7)$$

Dividing both sides of this equation by  $\sqrt{\alpha - E}$ , where  $\alpha$  is a parameter, integrating with respect to  $E$  from 0 to  $\alpha$  and putting  $U$  in place of  $\alpha$  (see [3] for detail) one gets

$$x_2(U) - x_1(U) = \frac{1}{\sqrt{2\pi}} \int_0^U \frac{T(E)dE}{\sqrt{U-E}}.$$

In the case when  $T(E) \equiv \tau$  that yields (6).

To prove that (6) is the sufficient condition of isochronicity we note that (6) implies

$$x'_2(U) - x'_1(U) = \frac{\sqrt{2}\tau}{2\pi\sqrt{U}}.$$

Substituting this expression into (7) and integrating we get  $T(E) \equiv \tau$ .  $\square$

As an immediate consequence we get the following proposition proved earlier in [7].

**Corollary 1** *If  $g(x) \in C^1(a, b)$  is an odd function, then the origin is an isochronous center iff  $g(x) = (2\pi/\tau)^2 x$ .*

In other words, if the potential (energy)  $U(x)$  is an even function of position  $x$  then the only isochronous system is the harmonic oscillator given above.

*Proof of theorem 1.* Let us suppose that the system (1) has an isochronous center. Then due to theorem 2 the relation (6) holds and we get

$$x_2(U) - \frac{\sqrt{2}\tau}{2\pi}\sqrt{U} = x_1 + \frac{\sqrt{2}\tau}{2\pi}\sqrt{U} \stackrel{def}{=} f(U).$$

Therefore

$$x'_2(U) = \frac{\tau}{2\sqrt{2\pi}\sqrt{U}} + f'(U), \tag{8}$$

$$x'_1(U) = \frac{-\tau}{2\sqrt{2\pi}\sqrt{U}} + f'(U). \tag{9}$$

Taking derivative in the both parts of (6) with respect to  $x$  we get for  $x < 0$

$$x'_2(U)U' - 1 = \frac{\sqrt{2}\tau}{2\pi\sqrt{U}}U'.$$

Therefore, using (8) we obtain

$$U' = \frac{2\pi}{\tau} \frac{-\sqrt{2U}}{1 - \frac{2\pi}{\tau}\sqrt{2U}f'(U)}. \tag{10}$$

Similarly, for  $x > 0$  we get from (9)

$$U' = \frac{2\pi}{\tau} \frac{\sqrt{2U}}{1 + \frac{2\pi}{\tau} \sqrt{2U} f'(U)}. \quad (11)$$

Therefore function  $g(x)$  can be expressed in the form (5).

Now it remains to show that

$$S(X) = \frac{2\pi}{\tau} X f'(X^2)$$

is a continuous function. Obviously, it is true if  $X \neq 0$ .

For  $X = x = 0$  we have the situation as follows. First note that (2) and (6) yield

$$U = \frac{2\pi^2}{\tau^2} x^2 + o(x^2).$$

Then for  $x, X > 0$  from (11) we get

$$S(X) = \frac{2\pi}{\tau} \sqrt{2U} f'(U) = \frac{\frac{2\pi^2}{\tau} \sqrt{2U}}{U'} - 1 = \frac{x \sqrt{1 + o(1)}}{x + o(x)} - 1.$$

Therefore

$$\lim_{X \rightarrow 0+} S(X) = 0.$$

For  $x, X < 0$  (10) yields

$$S(X) = -\frac{2\pi}{\tau} \sqrt{2U} f'(U) = -\frac{\frac{2\pi^2}{\tau} \sqrt{2U}}{U'} - 1 = -\frac{|x| \sqrt{1 + o(1)}}{x + o(x)} - 1.$$

It means  $\lim_{X \rightarrow 0-} S(X) = 0$  and, hence,  $S(X)$  is continuous at zero.

Let us prove that (5) is also the sufficient condition of isochronicity. For  $x > 0$  we can write (5) in the form

$$\frac{dU}{dx} = \frac{2\pi}{\tau} \frac{X}{1 + S(X)} = \frac{2\pi}{\tau} \frac{\sqrt{2U}}{1 + S(\sqrt{2U})}.$$

Integrating this equation we get

$$x_2(U) = \frac{\tau}{2\pi} (\sqrt{2U} + \int_0^{\sqrt{2U}} S(z) dz).$$

Similarly, for  $x < 0$  we obtain

$$x_1(U) = \frac{\tau}{2\pi}(-\sqrt{2U} + \int_0^{\sqrt{2U}} S(z)dz).$$

Due to the condition of the theorem  $S(z)$  is a continuous function, and, hence, the integral is convergent. Therefore (6) holds, i.e. the system has an isochronous center in the origin.  $\square$

In conclusion, we have proven that the Hamiltonian (3) has the isochronous center iff the condition (6) is satisfied. In case of a symmetric potential  $U(x)$  (even function of  $x$ ) the only solution is the harmonic oscillator. If  $U(x)$  is not symmetric (even), other solutions might be possible. However, for any polynomial  $U(x)$  (and  $g(x) = U'(x)$ ), the harmonic potential is still the only solution [1]. Thus, other nontrivial isochronic potentials can be invented by taking an analytic but not polynomial and not even function  $U(x)$ , in agreement with Urabe's criteria (5) of Theorem 1, which we have shown to be equivalent to (6). These criteria allow still for a quite large family of isochronous potentials  $U(x)$  and we can construct such potentials analytically. Indeed, differentiating the both sides of the equality (4) and taking into account (5) we get in the case of isochronous center

$$X \frac{dX}{dx} = g(x) = \frac{2\pi}{\tau} \frac{X}{1 + S(X)}.$$

Hence, we obtain the next formula, which for the first time appears in [7]

$$x = \frac{\tau}{2\pi} \int_0^X (1 + S(u)) du. \quad (12)$$

This formula together with (5) is a tool to construct isochronous potentials. Taking  $S(X) = X$  Urabe got

$$g(x) = \frac{2\pi}{\tau} [1 - (1 + \frac{4\pi}{\tau} x)^{-\frac{1}{2}}],$$

hence, the corresponding isochronous potential is

$$U(x) = 1 + \frac{2\pi}{\tau} x - \sqrt{1 + \frac{4\pi}{\tau} x}. \quad (13)$$

where  $-\frac{\tau}{4\pi} < x < \frac{3\tau}{4\pi}$ , i.e. the potential is an analytic function defined on a finite segment of real axis. Here, in the calculation, we have chosen the (negative) sign such that  $g(x=0) = 0$  is obeyed.

Let now

$$S(X) = \frac{2}{\pi} \operatorname{arctg} X.$$

Then (12) yields

$$x = \frac{\tau}{2\pi}X + \frac{\tau}{\pi^2}X \operatorname{arctg} X - \frac{\tau}{2\pi^2} \log(X^2 + 1).$$

Obviously,  $x(X)$  is strictly increasing on  $\mathbf{R}$  and  $x(0) = 0$ ,  $x(\mathbf{R}) = \mathbf{R}$ . Therefore,

$$g(x) = \frac{2\pi}{\tau} \frac{X(x)}{1 + \frac{2}{\pi} \operatorname{arctg}(X(x))}$$

is defined for all  $x \in \mathbf{R}$ , positive for  $x > 0$  and negative for  $x < 0$ . Hence, the corresponding potential  $U(x)$  is an analytic function defined on the whole real axis with the only one minimum in the origin. One can construct this potential at least in the form of power series. However, the potential is not an entire function. As we have mentioned above it was shown in [1] (in fact, it is an immediate consequence of formula (6)), that the only polynomial isochronous potential is the quadratic one. We also see that there are analytic potentials defined on whole real axis. Thus the question naturally arises whether there are isochronous potentials defined by entire functions? Another still open and interesting question is the investigation of the isochronicity property of non-monotonic potentials.

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